On the Negative Pell Equation $y^2 = 7x^2 - 14$

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Abstract – The binary quadratic Diophantine equation represented by the negative pellian $y^2 = 7x^2 - 14$ is analyzed for its non-zero distinct solutions. A few interesting relations among the solutions are given. Further, employing the solutions of the above hyperbola, we have obtained solutions of other choices of hyperbolas, parabolas and Pythagorean triangle.

Index Terms – Binary quadratic, Hyperbola, Parabola, Integral solutions, Pell equation, Mathematics subject Classification(2010):11D09.

1. INTRODUCTION

The binary quadratic equations of the form $y^2 = Dx^2 + 1$ where D is non-square positive integer has been selected by various mathematicians for its non-trivial integer solutions when D takes different integral values[1-4].For an extensive review of various problems, one may refer[5-14]. In this communication, yet another interesting equation given by $y^2 = 7x^2 - 14$ is considered and infinitely many integer solutions are obtained. A few interesting properties among the solutions are presented.

2. METHOD OF ANALYSIS

The negative Pell equation representing hyperbola under consideration is,

$$y^2 = 7x^2 - 14 \qquad (1)$$

The smallest positive integer solutions of (1) are,

$$x_0 = 3, y_0 = 7$$

The general solution (x_n, y_n) of (1) is given by,

$$\tilde{x}_n = \frac{1}{2\sqrt{7}} g_n \;, \; \tilde{y}_n = \frac{1}{2} f_n$$

Where,

$$f_n = (8+3\sqrt{7})^{n+1} + (8-3\sqrt{7})^{n+1}$$

$$g_n = (8+3\sqrt{7})^{n+1} - (8-3\sqrt{7})^{n+1}$$

Applying Brahmagupta lemma between (x_0, y_0) and $(\tilde{x}_n, \tilde{y}_n)$ the other integer solution of (1) are given by,

$$x_{n+1} = \frac{3}{2} f_n + \frac{7}{2\sqrt{7}} g_n$$

$$y_{n+1} = \frac{7}{2} f_n + \frac{21}{2\sqrt{7}} g_n$$

The recurrence relation satisfied by the solution x and y are given by,

$$x_{n+3} - 16x_{n+2} + x_{n+1} = 0$$

$$y_{n+3} - 16y_{n+2} + y_{n+1} = 0$$

Some numerical examples of x and y satisfying (1) are given in the Table 1 below,

Table 1: Examples

n	x_n	y_n
0	3	7
1	45	119
2	717	1897
3	11427	30233
4	182115	481831

From the above table, we observe some interesting relations among the solutions which are presented below.

- Both x_n *and* y_n values are odd .
- Each of the following expression is a nasty number:
- $[12 + 18x_{2n+2} - 6y_{2n+2}]$

- $\frac{1}{21}\left[252+714x_{2n+2}-42x_{2n+3}\right]$
- $\frac{1}{4}\left[48+135x_{2n+2}-3y_{2n+3}\right]$
- $\frac{1}{8}\left[96+271x_{2n+2}-x_{2n+4}\right]$
- $\frac{1}{127}\left[1524+4302x_{2n+2}-6y_{2n+4}\right]$
- $\frac{1}{28}\left[336+63x_{2n+3}-357y_{2n+2}\right]$
- $\frac{1}{7}\left[84+6y_{2n+3}-90y_{2n+2}\right]$
- $\frac{1}{56}\left[672+3y_{2n+4}-717y_{2n+2}\right]$
- $\frac{1}{7}\left[84+1890x_{2n+3}-714y_{2n+3}\right]$
- $\frac{1}{4}\left[48+2151x_{2n+3}-51y_{2n+4}\right]$
- $\frac{1}{4}\left[48+135x_{2n+4}-813y_{2n+3}\right]$
- $\frac{1}{7}\left[84+90y_{2n+4}-1434y_{2n+3}\right]$
- $\frac{1}{7}\left[84+3794x_{2n+3}-238x_{2n+4}\right]$
- $\frac{1}{7}\left[84+3014x_{2n+4}-11382y_{2n+4}\right]$

Each of the following expressions is a cubical integer.

- $3x_{3n+3}-y_{3n+3}+9x_{n+1}-3y_{n+1}$
- $9\left[17x_{3n+3}-x_{3n+4}+51x_{n+1}-3x_{n+2}\right]$
- $64\left[45x_{3n+3}-y_{3n+4}+135x_{n+1}-3y_{n+2}\right]$
- $16129\left[717x_{3n+3}-y_{3n+5}+2151x_{n+1}-3y_{n+3}\right]$
- $3136\left[21x_{3n+4}-119y_{3n+3}+63x_{n+2}-357y_{n+1}\right]$
- $12544\left[y_{3n+5}-239y_{3n+3}+3y_{n+3}-717y_{n+1}\right]$
- $49\left[315x_{3n+4}-119y_{3n+4}+945x_{n+2}-357y_{n+2}\right]$
- $3136\left[5019x_{3n+4}-119y_{3n+5}+15057x_{n+2}-357y_{n+3}\right]$
- $441\left[1897x_{3n+4}-119x_{3n+5}+5691x_{n+2}-357x_{n+3}\right]$
- $64\left[45x_{3n+5}-271y_{3n+4}+135x_{n+3}-813y_{n+2}\right]$
- $441\left[45y_{3n+5}-717y_{3n+4}+135y_{n+3}-2151y_{n+2}\right]$
- $49\left[y_{3n+4}-15y_{3n+3}+3y_{n+2}-45y_{n+1}\right]$
- $717x_{3n+5}-271y_{3n+5}+2151x_{n+3}-813y_{n+3}$
- $2304\left[271x_{3n+3}-x_{3n+5}+813x_{n+1}-3x_{n+3}\right]$

Each of the following expressions is a biquadratic integer.

- $3x_{4n+4}-y_{4n+4}+12x_{2n+2}-4y_{2n+2}+6$
- $27\left[17x_{4n+4}-x_{4n+5}+68x_{2n+2}-4x_{2n+3}+18\right]$
- $512\left[45x_{4n+4}-y_{4n+5}+180x_{2n+2}-4y_{2n+3}+48\right]$
- $110592\left[271x_{4n+4}-x_{4n+6}+1084x_{2n+2}-4x_{2n+4}+288\right]$
- $2048383\left[717x_{4n+4}-y_{4n+6}+2868x_{2n+2}-4y_{2n+4}+762\right]$
- $175616\left[21x_{4n+5}-119y_{4n+4}+84x_{2n+3}-476y_{2n+2}+336\right]$
- $1404928\left[y_{4n+6}-239y_{4n+4}+4y_{2n+4}-956y_{2n+2}+672\right]$
- $343\left[315x_{4n+5}-119y_{4n+5}+1260x_{2n+3}-476y_{2n+3}+42\right]$
- $175616\left[5019x_{4n+5}-119y_{4n+6}+20076x_{2n+3}-476y_{2n+4}+336\right]$
- $9261\left[1897x_{4n+5}-119x_{4n+6}+7588x_{2n+3}-476x_{2n+4}+126\right]$
- $512\left[45x_{4n+6}-271y_{4n+5}+180x_{2n+4}-1084y_{2n+3}+48\right]$
- $9261\left[45y_{4n+6}-717y_{4n+5}+180y_{2n+4}-2868y_{2n+3}+126\right]$
- $343\left[y_{4n+5}-15y_{4n+4}+4y_{2n+3}-60y_{2n+2}+42\right]$
- $717x_{4n+6}-271y_{4n+6}+2868x_{2n+4}-1084y_{2n+4}+6$

- Relations among the solutions are given below:

- $8x_{n+1}=x_{n+2}-3y_{n+1}$

- $21x_{n+1} = y_{n+2} - 8y_{n+1}$
- $336x_{n+1} = y_{n+3} - 127y_{n+1}$
- $x_{n+1} = 8x_{n+2} - 3y_{n+2}$
- $x_{n+1} = 16x_{n+2} - x_{n+3}$
- $8x_{n+1} = 127x_{n+2} - 3y_{n+3}$
- $x_{n+1} = x_{n+3} - 6y_{n+2}$
- $21x_{n+1} = 8y_{n+3} - 127y_{n+2}$
- $x_{n+1} = 127x_{n+3} - 48y_{n+3}$
- $8x_{n+2} = x_{n+1} + 3y_{n+2}$
- $127x_{n+2} = 3y_{n+3} + 8x_{n+3}$
- $21x_{n+2} = 8y_{n+2} - y_{n+1}$
- $8x_{n+2} = x_{n+3} - 3y_{n+2}$
- $21x_{n+2} = y_{n+3} - 8y_{n+2}$
- $x_{n+2} = 8x_{n+3} - 3y_{n+3}$
- $3x_{n+3} = 48x_{n+2} - 3x_{n+1}$
- $21x_{n+3} = 127y_{n+2} - 8y_{n+1}$
- $21x_{n+3} = 8y_{n+3} - y_{n+2}$
- $3y_{n+1} = x_{n+2} - 8x_{n+1}$
- $48y_{n+1} = x_{n+3} - 127x_{n+1}$
- $y_{n+1} = 8y_{n+2} - 21x_{n+2}$
- $3y_{n+1} = 8x_{n+3} - 127x_{n+2}$
- $y_{n+1} = y_{n+3} - 42x_{n+2}$
- $8y_{n+1} = 127y_{n+2} - 21x_{n+3}$
- $y_{n+1} = 16y_{n+2} - y_{n+3}$
- $y_{n+1} = 127y_{n+3} - 336x_{n+3}$
- $3y_{n+2} = 8x_{n+2} - x_{n+1}$
- $6y_{n+2} = x_{n+3} - x_{n+1}$
- $127y_{n+2} = 8y_{n+3} - 21x_{n+1}$
- $3y_{n+2} = x_{n+3} - 8x_{n+2}$
- $8y_{n+2} = y_{n+3} - 21x_{n+2}$
- $y_{n+2} = 8y_{n+3} - 21x_{n+3}$
- $3y_{n+3} = 127x_{n+2} - 8x_{n+1}$
- $48y_{n+3} = 127x_{n+3} - x_{n+1}$
- $y_{n+3} = 16y_{n+2} - y_{n+1}$
- $3y_{n+3} = 8x_{n+3} - x_{n+2}$

3. REMARKABLE OBSERVATIONS

Table 2: Hyperbola

Hyperbola	**(X,Y)**
$Y^2 - 7X^2 = 784$	$(6y_{n+1} - 14x_{n+1}, 42x_{n+1} - 14y_{n+1})$
$Y^2 - 7X^2 = 1764$	$(3x_{n+2} - 45x_{n+1}, 119x_{n+1} - 7x_{n+2})$
$Y^2 - 7X^2 = 50176$	$(6y_{n+2} - 238x_{n+1}, 630x_{n+1} - 14y_{n+2})$
$Y^2 - 7X^2 = 1806336$	$(6x_{n+3} - 1434x_{n+1}, 3794x_{n+1} - 14x_{n+3})$
$Y^2 - 7X^2 = 12645136$	$(6y_{n+3} - 3794x_{n+1}, 10038x_{n+1} - 14y_{n+3})$
$Y^2 - 7X^2 = 50176$	$(90y_{n+1} - 14x_{n+2}, 42x_{n+2} - 238y_{n+1})$
$Y^2 - 7X^2 = 7056$	$(34y_{n+1} - 2y_{n+2}, 6y_{n+2} - 90y_{n+1})$
$Y^2 - 7X^2 = 1806336$	$(542y_{n+1} - 2y_{n+3}, 6y_{n+3} - 1434y_{n+1})$
$Y^2 - 7X^2 = 784$	$(90y_{n+2} - 238x_{n+2}, 630x_{n+2} - 238y_{n+2})$
$Y^2 - 7X^2 = 7056$	$(90x_{n+3} - 1434x_{n+2}, 3794x_{n+2} - 238x_{n+3})$
$Y^2 - 7X^2 = 50176$	$(90y_{n+3} - 3794x_{n+2}, 10038x_{n+2} - 238y_{n+3})$
$Y^2 - 7X^2 = 50176$	$(1434y_{n+2} - 238x_{n+3}, 630x_{n+3} - 3794y_{n+2})$
$Y^2 - 7X^2 = 345744$	$(3794y_{n+2} - 238y_{n+3}, 630y_{n+3} - 10038y_{n+2})$
$Y^2 - 7X^2 = 784$	$(1434y_{n+3} - 3794x_{n+3}, 10038x_{n+3} - 3794y_{n+3})$

Employing linear combination among the solutions of (1), one may generate integer solutions for other choices of parabolas which are presented in table 3 below:

Parabola	**(X,Y)**
$14Y-7X^2=392$	$(6y_{n+1}-14x_{n+1}, 42x_{2n+2}-14y_{2n+2})$
$21Y-7X^2=882$	$(3x_{n+2}-45x_{n+1}, 119x_{2n+2}-7x_{2n+3})$
$112Y-7X^2=25088$	$(6y_{n+2}-238x_{n+1}, 630x_{2n+2}-14y_{2n+3})$
$672Y-7X^2=903168$	$(6x_{n+3}-1434x_{n+1}, 3794x_{2n+2}-14x_{2n+4})$
$1778Y-7X^2=6322568$	$(6y_{n+3}-3794x_{n+1}, 10038x_{2n+2}-14y_{2n+4})$
$112Y-7X^2=25088$	$(90y_{n+1}-14x_{n+2}, 42x_{2n+3}-238y_{2n+2})$
$42Y-7X^2=3528$	$(34y_{n+1}-2y_{n+2}, 6y_{2n+3}-90y_{2n+2})$
$672Y-7X^2=903168$	$(542y_{n+1}-2y_{n+3}, 6y_{2n+4}-1434y_{2n+2})$
$14Y-7X^2=392$	$(90y_{n+2}-238x_{n+2}, 630x_{2n+3}-238y_{2n+3})$
$42Y-7X^2=3528$	$(90x_{n+3}-1434x_{n+2}, 3794x_{2n+3}-238x_{2n+4})$
$112Y-7X^2=25088$	$(90y_{n+3}-3794x_{n+2}, 10038x_{2n+3}-238y_{2n+4})$
$112Y-7X^2=25088$	$(1434y_{n+2}-238x_{n+3}, 630x_{2n+4}-3794y_{2n+3})$
$294Y-7X^2=172872$	$(3794y_{n+2}-238y_{n+3}, 630y_{2n+4}-10038y_{2n+3})$
$14Y-7X^2=392$	$(1434y_{n+3}-3794x_{n+3}, 10038x_{2n+4}-3794y_{2n+4})$

Table 3: Parabola

4. CONCLUSION

Conclusion part depicts the main points as the constructive f In this paper, we have presented infinitely many integer solutions for the negative Pell Equations $y^2=7x^2-14$. As the binary quadratic Diophantine equations are rich in variety, one may search for the other choices of Pell Equations and determine their integer solutions along with suitable properties.

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